On the modeling of drift fluxes with self-consistent electric field in the SOLPS code.

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in collaboration with M. Restelli[1], D. Coster[1], E. Sonnendrücker[1], T. Feher[1] and Juan Vicente Gutierrez Santacreu[2]

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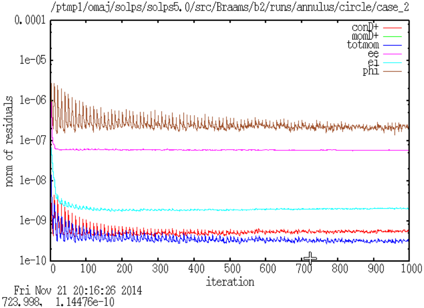
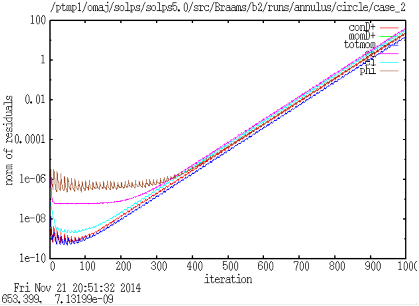
Motivations:
Time-step limitations in SOLPS with drifts

Test case by David Coster - circular geometry (closed field lines).





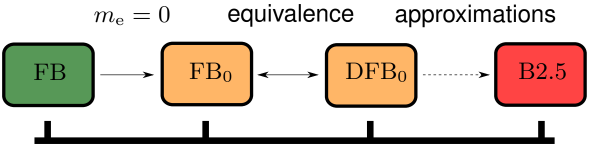
Time step $\Delta t = 5.0 \times 10^{-7}$s.

Time step $\Delta t = 8.0 \times 10^{-7}$s.

- Without drift physics, one observes fast convergence with $\Delta t = 10^{-3}$s.
- We aim to develop a stand-alone code as a test-bed for models and schemes.
- This motivates our study of the mathematical structure of the B2.5 model [Rozhansky et al. Nucl. Fus. (2001)].

Outline

- **The self-consistent electric potential in a hierarchy of multi-fluid models:**



- FB Full system of Braginskii equations
- FB_0 Quasi-neutral zero-electron-mass limit of FB
- DFB_0 Drift formulation equivalent to FB_0
- $\mathrm{B2.5}$ Model of the B2.5 code

- **Initial development of a test code (Marco Restelli):**
  - exploring numerical methods for this family of fluid models;
  - addressing basic issues (sensitivity to input data, noisy sources, etc ...).

The self-consistent electric potential in a hierarchy of multi-fluid models.

General idea:

- Anomalous fluxes are added *ad hoc* later: no derivation.
- Neutrals can enter either as sources or fluid species.
- Divergence-free terms are kept in the equation for clarity.
- Refrain from approximations as far as possible.
- Emphasis on the mathematical structure of the equations and possible sources of the instability.

Full system of Braginskii equations (FB)

Model equations

- Main equations of the model: for $\alpha \in \mathrm{Sp} = \{$various ion species, electrons$\}$,

$$\begin{cases} \partial_t n_\alpha + \nabla \cdot \big(n_\alpha v_\alpha\big) = S_{n,\alpha}, \\ \partial_t (m_\alpha n_\alpha v_\alpha) + \nabla \cdot \big(m_\alpha n_\alpha v_\alpha \otimes v_\alpha + \pi_\alpha\big) \\ \qquad\qquad\qquad = -\nabla p_\alpha + e_\alpha n_\alpha \big(-\nabla\phi + \frac{v_\alpha \times B}{c}\big) + R_\alpha + S_{M,\alpha}, \\ \partial_t \big(\tfrac{3}{2} p_\alpha\big) + \nabla \cdot \big(\tfrac{3}{2} p_\alpha v_\alpha + q_\alpha\big) + p_\alpha \nabla \cdot v_\alpha + \pi_\alpha : \nabla v_\alpha = Q_\alpha + S_{T,\alpha}, \\ \nabla \cdot J = 0, \end{cases}$$

  where $J = \sum_{\alpha\in\mathrm{Sp}} e_\alpha n_\alpha v_\alpha$ and $S_{n,\alpha}$, $S_{M,\alpha}$, and $S_{T,\alpha}$ are sources.
- Closure relations for π_α, R_α, Q_α, and q_α, satisfying

$$\sum_{\alpha\in\mathrm{Sp}} R_\alpha = 0, \quad \sum_{\alpha\in\mathrm{Sp}} \big(Q_\alpha + v_\alpha \cdot R_\alpha\big) = 0.$$

- The electric potential ϕ is determined as a Lagrange multiplier for $\nabla \cdot J = 0$.
- Without sources and with appropriate boundary conditions, energy is conserved:

$$W(t) = \sum_{\alpha\in\mathrm{Sp}} \int_\Omega \big(\frac{1}{2} m_\alpha n_\alpha |v_\alpha|^2 + \frac{3}{2} p_\alpha\big) dV = \text{constant},$$

  as a consequence of $\nabla \cdot J = 0$.

Braginskii equations with zero electron mass (FB_0)

Model equations

- Main equations of the model: for $a \in \mathrm{Sp}_0 = \mathrm{Sp} \setminus \{\mathrm{e}\}$

$$\begin{cases} \partial_t n_a + \nabla \cdot \big(n_a v_a\big) = S_{n,a}, \\ \partial_t \big(m_a n_a v_a\big) + \nabla \cdot \big(m_a n_a v_a \otimes v_a + \pi_a\big) \\ \qquad\qquad\qquad = -\nabla p_a + e_a n_a\big(-\nabla\phi + \frac{v_a \times B}{c}\big) + R_a + S_{M,a}, \\ \partial_t \big(\frac{3}{2}p_a\big) + \nabla \cdot \big(\frac{3}{2}p_a v_a + q_a\big) + p_a \nabla \cdot v_a + \pi_a : \nabla v_a = Q_a + S_{T,a}, \\ \nabla \cdot J = 0, \end{cases}$$

$$\begin{cases} n_\mathrm{e} = \sum\limits_{a \in \mathrm{Sp}_0} Z_a n_a, \\ 0 = -\nabla p_\mathrm{e} - e n_\mathrm{e}\big(-\nabla\phi + \frac{v_\mathrm{e} \times B}{c}\big) + R_\mathrm{e} + S_{M,\mathrm{e}}, \\ \partial_t \big(\frac{3}{2}p_\mathrm{e}\big) + \nabla \cdot \big(\frac{3}{2}p_\mathrm{e} v_\mathrm{e} + q_\mathrm{e}\big) + p_\mathrm{e} \nabla \cdot v_\mathrm{e} = Q_\mathrm{e} + S_{T,\mathrm{e}}, \quad (\pi_\mathrm{e} = 0). \end{cases}$$

- Again, ϕ is a Lagrange multiplier for $\nabla \cdot J = 0$.
- Again, the relevant energy is conserved

$$W(t) = \int_\Omega \Big[\sum_{a \in \mathrm{Sp}_0} \big(\frac{1}{2} m_a n_a |v_a|^2 + \frac{3}{2} p_a\big) + \frac{3}{2} p_\mathrm{e}\Big] dV = \text{constant},$$

$\Rightarrow$ Energy conserving schemes derived by Juan Vicente Gutierrez Santacreu.

Braginskii equations with zero electron mass (FB_0)

Determining the electron velocity

- The electron velocity should be fully determined by the electron force balance

$$0 = -\nabla p_{\mathrm{e}} - en_{\mathrm{e}}\big(-\nabla\phi + \tfrac{v_{\mathrm{e}}\times B}{c}\big) + R_{\mathrm{e}} + S_{M,\mathrm{e}}.$$

- The friction force R_{e} on electrons is crucial:

$$R_{\mathrm{e}} = en_{\mathrm{e}}(J_{\parallel}/\sigma_{\parallel} + J_{\perp}/\sigma_{\perp}) + R_T.$$

Proposition

Given sufficiently regular $n_{\mathrm{e}}, p_{\mathrm{e}}, n_a, v_a$ *for* $a \in \mathrm{Sp}_0$ *and* ϕ*, with both* $n_{\mathrm{e}}, 1/n_{\mathrm{e}} \in L^{\infty}$*, the electron force balance is equivalent to*

$$J = \hat{\sigma}\big[-\nabla\phi + E_T\big],$$

where $\hat{\sigma}$ *is bounded and positive-definite, and* E_T *is a function of* $n_{\mathrm{e}}, \nabla p_{\mathrm{e}}, R_T, n_a, v_a$,

$$en_{\mathrm{e}}E_T = \nabla p_{\mathrm{e}} - R_T - S_{M,\mathrm{e}} + \sum_{a\in\mathrm{Sp}_0} e_a n_a v_a \times B/c,$$

and thus the electron velocity determined by $en_{\mathrm{e}}v_{\mathrm{e}} = \sum_a e_a n_a v_a - J$.

Equivalent form of the perpendicular current density

- Total momentum balance equation for the model FB_0:

$$\partial_t \mathcal{P} + \nabla \cdot \mathcal{S} = -\nabla p + J \times B/c + S_M,$$

with momentum/momentum-flux pair

$$\mathcal{P} = \sum_{a \in \mathrm{Sp}_0} m_a n_a v_a, \quad \mathcal{S} = \sum_{a \in \mathrm{Sp}_0} m_a n_a v_a \otimes v_a + \pi_a,$$

where $p = \sum_a p_a + p_\mathrm{e}$ and $S_M = \sum_a S_{M,a} + S_{M,e}$.

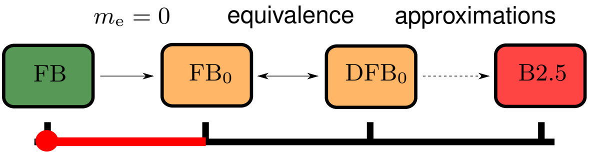
- We can solve for $J_\perp$ from the total momentum balance:

$$J_\perp = \underbrace{\frac{c}{|B|} b \times \nabla p}_{\text{diamagnetic current}} + \frac{c}{|B|} b \times \Big[\partial_t \mathcal{P} + \nabla \cdot \mathcal{S} - S_M\Big].$$

Proposition

For a sufficiently regular solution,

$$J_\perp = \big[\hat{\sigma}(-\nabla\phi + E_T)\big]_\perp = \frac{c}{|B|} b \times \nabla p + \frac{c}{|B|} b \times \Big[\partial_t \mathcal{P} + \nabla \cdot \mathcal{S} - S_M\Big].$$



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Drift velocity

Basic ideas: a simple textbook example

- Euler's equation for an electrically charged fluid:

$$mn\frac{du}{dt} = -\nabla p + en\Big(E + \frac{u \times B}{c}\Big)$$

- The perpendicular part can be rewritten by solving for $u_\perp$:

$$\begin{aligned} u_\perp &= c\frac{E \times B}{B^2} + c\frac{B \times \nabla p}{enB^2} - \frac{mc}{eB^2} B \times \frac{du}{dt} \\ &= \underbrace{c\frac{E \times B}{B^2}}_{E \times B \text{ drift}} + \underbrace{c\frac{B \times \nabla p}{enB^2}}_{\text{diamagnetic drift } u_*} - \underbrace{\frac{1}{\omega_c} b \times \frac{du}{dt}}_{\text{inertia}} \end{aligned}$$

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initial guess $u_{0\perp}$ for fixed-point iterations.

- This turns an initial/boundary value problem into a fixed-point problem.

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initial guess $u_{0\perp}$ for fixed-point iterations.

- This turns an initial/boundary value problem into a fixed-point problem.
- Guiding-center drifts:

$$\nabla \cdot (nu_*) = \nabla \cdot \Big[-\frac{cp}{e}B \times \nabla\frac{1}{B^2}\Big] + \frac{c}{e}\nabla\frac{p}{B^2} \cdot \nabla \times B.$$

Drift velocity

Drift formulation of the momentum balance equation

- Define the current densities

$$J_a^{(\mathrm{r})} = \frac{c}{|B|} b \times \big[\partial_t(m_a n_a v_a) + \nabla \cdot (m_a n_a v_a \otimes v_a + \pi_a) - S_{M,a}\big],$$

$$J_\perp^{(\mathrm{r})} = \sum_{a \in \mathrm{Sp}_0} J_a^{(\mathrm{r})} = J_\perp - \frac{cB \times \nabla p}{B^2},$$

where $\{v_a\}_a \mapsto J_a^{(\mathrm{r})}$ is a linear operator acting on velocities $\{v_a\}_a$.

Proposition

For every ion species a, the momentum equation

$$\partial_t(m_a n_a v_a) + \nabla \cdot \big(m_a n_a v_a \otimes v_a + \pi_a\big) = -\nabla p_a + e_a n_a\big(-\nabla\phi + \tfrac{v_a \times B}{c}\big) + R_a + S_{M,a},$$

is formally equivalent to the parallel momentum balance complemented with the fixed-point problem

$$v_{a\perp} = c\frac{B \times \nabla\phi}{B^2} + c\frac{B \times \nabla p_a}{e_a n_a B^2} - \frac{D_a}{T_a + Z_a T_\mathrm{e}}\left[\frac{\nabla_\perp p}{n_\mathrm{e}} - \frac{3}{2}\nabla_\perp T_\mathrm{e}\right] + \frac{J_a^{(\mathrm{r})}}{e_a n_a} + \frac{1}{\tau_\mathrm{e}\omega_{ce}}\frac{b \times J_\perp^{(\mathrm{r})}}{e n_\mathrm{e}},$$

where $D_a = \frac{1}{\tau_\mathrm{e}\omega_{ce}}\frac{c}{e|B|}(T_a + Z_a T_\mathrm{e})$ is the collisional diffusion coefficient.

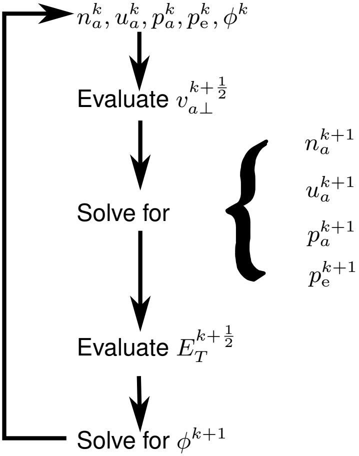
- Definition of the parallel velocity $u_a = b \cdot v_a$.
- Main equations of the model: for $a \in \mathrm{Sp}_0 = \mathrm{Sp} \setminus \{\mathrm{e}\}$

$$
\begin{cases}
\partial_t n_a + \nabla \cdot \big(n_a(u_a b + v_{a\perp})\big) = S_{n,a}, \\
\partial_t(m_a n_a u_a) + \nabla \cdot \big(m_a n_a u_a(u_a b + v_{a\perp}) + \pi_a \cdot b\big) - m_a n_a(u_a b + v_{a\perp}) \cdot \nabla b \cdot v_{a\perp} \\
\qquad\qquad - \pi_a : \nabla b = -b \cdot \nabla p_a - e_a n_a b \cdot \nabla \phi + b \cdot R_a + b \cdot S_{M,a}, \\
v_{a\perp} = c\frac{B \times \nabla \phi}{B^2} + c\frac{B \times \nabla p_a}{e_a n_a B^2} - \frac{D_a}{T_a + Z_a T_\mathrm{e}}\Big[\frac{\nabla_\perp p}{n_\mathrm{e}} - \frac{3}{2}\nabla_\perp T_\mathrm{e}\Big] + \frac{J_a^{(\mathrm{r})}}{e_a n_a} + \frac{1}{\tau_\mathrm{e}\omega_{ce}}\frac{b \times J_\perp^{(\mathrm{r})}}{e n_\mathrm{e}}, \\
\partial_t\big(\tfrac{3}{2}p_a\big) + \nabla \cdot \big(\tfrac{3}{2}p_a v_a + q_a\big) + p_a \nabla \cdot v_a + \pi_a : \nabla v_a = Q_a + S_{T,a}, \\
\partial_t\big(\tfrac{3}{2}p_\mathrm{e}\big) + \nabla \cdot \big(\tfrac{3}{2}p_\mathrm{e} v_\mathrm{e} + q_\mathrm{e}\big) + p_\mathrm{e} \nabla \cdot v_\mathrm{e} = Q_\mathrm{e} + S_{T,\mathrm{e}}, \\
\nabla \cdot \big[\hat{\sigma}(\nabla \phi - E_T)\big] = 0,
\end{cases}
$$

- Quasi-neutrality is implied.
- Both J and v_e are given explicitly as functions of the other variables.

Drift formulation of Braginskii equations (DFB_0)

Tentative iterative scheme



Drift formulation of Braginskii equations (DFB_0)

Choice of the equation for the potential

- Two equivalent forms of the perpendicular current

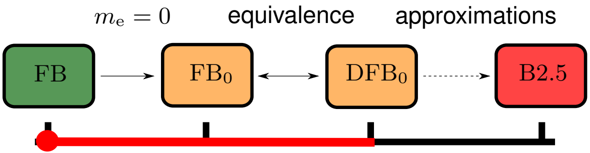
$$J_\perp = \left[\hat{\sigma}(-\nabla\phi + E_T)\right]_\perp = \frac{c}{|B|} b \times \nabla p + \frac{c}{|B|} b \times \left[\partial_t \mathcal{P} + \nabla \cdot \mathcal{S} - S_M\right].$$

- Two corresponding forms of the potential equation $\nabla \cdot J = 0$:

$$\nabla \cdot \left[\hat{\sigma}(\nabla\phi^{k+1} - E_T^{k+\frac{1}{2}})\right] = 0, \qquad \text{(well-posed)},$$

$$\nabla \cdot \left[\sigma_\parallel \nabla_\parallel \phi^{k+1} - \tilde{J}^{k+\frac{1}{2}}\right] = 0, \qquad \text{(ill-posed)}.$$

- In addition, E_T does not involve the time derivative of v_a.
- However, in the “well-posed form” ambipolar terms are not automatically canceled.
- Coupling the potential equation with the electron pressure gradients?



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Approximated model in the B2.5 code (B2.5)

Model equations

- Main equations of the model: for $a \in \mathrm{Sp}_0 = \mathrm{Sp} \setminus \{\mathrm{e}\}$

$$
\begin{cases}
\partial_t n_a + \nabla \cdot \big(n_a(u_a b + v_{a\perp})\big) = S_{n,a}, \\
\partial_t (m_a n_a u_a) + \nabla \cdot \big(m_a n_a u_a (u_a b + v_{a\perp}) + \pi_a \cdot b\big) - m_a n_a (u_a b + v_{a\perp}) \cdot \nabla b \cdot v_{a\perp} \\
\qquad - \pi_a : \nabla b = -b \cdot \nabla p_a - e_a n_a b \cdot \nabla \phi + b \cdot R_a + b \cdot S_{M,a}, \\
v_{a\perp} = c \frac{B \times \nabla \phi}{B^2} + c \frac{B \times \nabla p_a}{e_a n_a B^2} - \frac{D_a}{T_a + Z_a T_\mathrm{e}} \Big[\frac{\nabla_\perp p}{n_\mathrm{e}} - \frac{3}{2} \nabla_\perp T_\mathrm{e} \Big] + \frac{J_\perp^{(\mathrm{r})}}{e n_\mathrm{e}}, \\
\partial_t \big(\tfrac{3}{2} p_a\big) + \nabla \cdot \big(\tfrac{3}{2} p_a v_a + q_a\big) + p_a \nabla \cdot v_a + \pi_a : \nabla v_a = Q_a + S_{T,a}, \\
\partial_t \big(\tfrac{3}{2} p_\mathrm{e}\big) + \nabla \cdot \big(\tfrac{3}{2} p_\mathrm{e} v_\mathrm{e} + q_\mathrm{e}\big) + p_\mathrm{e} \nabla \cdot v_\mathrm{e} = Q_\mathrm{e} + S_{T,\mathrm{e}}, \\
\nabla \cdot \big[\sigma_\parallel \nabla_\parallel \phi - \tilde{J}\big] = 0.
\end{cases}
$$

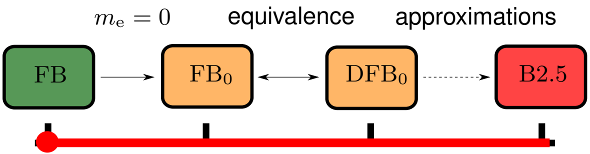
  This reduces to the model by Rozhansky et al. [Nucl. Fus. (2001)].

- Approximations in the drift velocity (break energy conservation):

$$
\underbrace{\frac{J_a^{(\mathrm{r})}}{e_a n_a} \approx \frac{J_\perp^{(\mathrm{r})}}{e n_\mathrm{e}}}_{\text{single ion species}}, \qquad \underbrace{\frac{1}{\tau_\mathrm{e} \omega_{ce}} \frac{b \times J_\perp^{(\mathrm{r})}}{e n_\mathrm{e}}}_{\text{higher-order term neglected}} \approx 0 \,.
$$

- Remark: The potential equation in the iterative scheme becomes ill-defined.

A few conclusions from the theory



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- Relationship among a class of multi-fluid models.
- An extended version of the B2.5 model, which is equivalent to Braginskii equations with zero electron mass and quasi-neutrality.
- Two equivalent forms of the potential equation.
- Possible sources of the step-size limitations in B2.5:
  - Ill-defined boundary value problem for the potential.
  - Fixed point iteration.

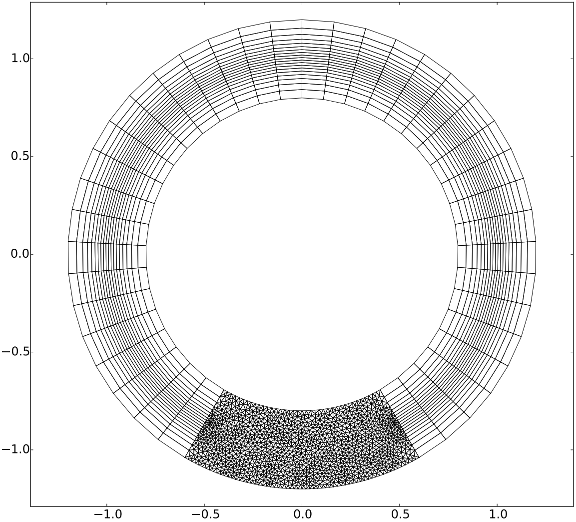
Initial development of a new code

Concept of the new code (by Marco Restelli)

- A code for multi-fluid models of the plasma edge which
  - provides a test-bed for numerical schemes;
  - allows us to explore the various formulations and models;
  - possibly includes uncertainty quantification (perturbation of data, noise, ...).
- Geometry:
  - Unstructured grid in generic coordinates (avoid local coordinate patches).
  - Possibility of flux-surface alignment and grid refinement.
- Space discretization:
  - Finite element method on unstructured grids.
  - SUPG stabilization of the convective terms.
  - MPI parallelization with domain decomposition.
  - MUMPS/PASTIX/PETSC/... libraries to solve the linear systems.
- Time discretization:
  - Standard diagonally implicit;
  - Energy-conserving slitting schemes (Juan Vicente Gutierrez Santacreu).

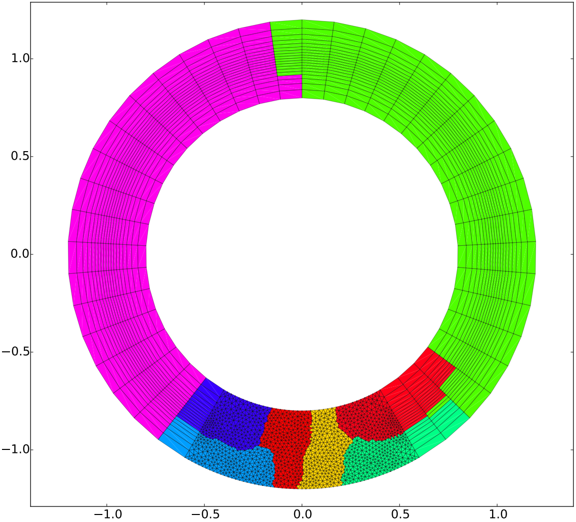
Initial development of a new code

Example of grid in circular geometry (by Marco Restelli)



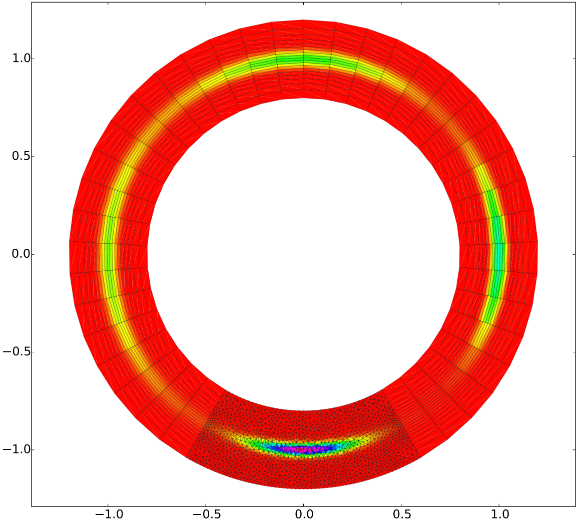
Initial development of a new code

Partition of the grid (by Marco Restelli)



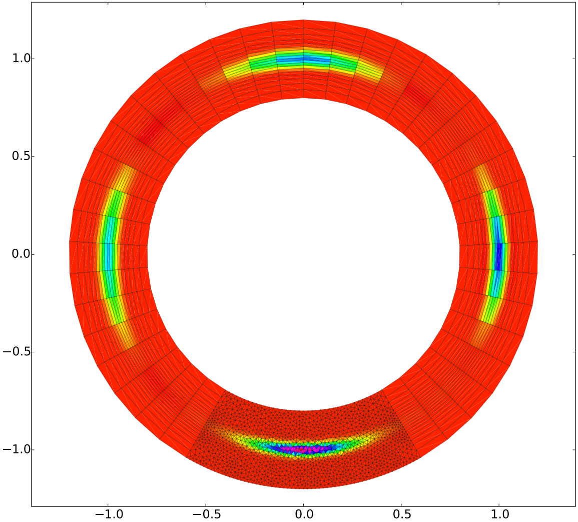
Initial development of a new code

Parallel advection - first-order upwind stabilization (by Marco Restelli)



Initial development of a new code

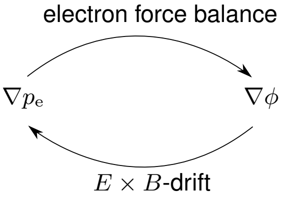
Parallel advection - second-order upwind stabilization (by Marco Restelli)



Open discussion:

Testing the coupling between the potential and the electron pressure gradient

- Sketch of the coupling:



- Minimal set of equations:

$$\begin{cases} \partial_t n + \nabla \cdot \big(n v_E - D \nabla_\perp n \big) = S, \\ \partial_t \frac{3}{2} p_\mathrm{e} + \nabla \cdot \big[\frac{3}{2} p_\mathrm{e} (v_E - J/(e n_\mathrm{e})) \big] + p_\mathrm{e} \nabla \cdot \big(v_E - J/(e n_\mathrm{e}) \big) = 0, \\ \nabla \cdot J = 0, \quad J = -\sigma_\parallel \nabla_\parallel \phi - \sigma_\mathrm{reg} \nabla_\perp \phi + \hat{\sigma} (\nabla p_\mathrm{e} - R_T)/(e n_\mathrm{e}), \end{cases}$$

where S is a possibly noisy source, $n_\mathrm{e} = Zn$, $p_\mathrm{e} = n_\mathrm{e} T_\mathrm{e}$ and

$$v_E = c \frac{B \times \nabla \phi}{B^2}, \qquad R_T = -0.71 n_\mathrm{e} \nabla_\parallel T_\mathrm{e} - \frac{3 n_\mathrm{e}}{2 \omega_{ce} \tau_\mathrm{e}} b \times \nabla T_\mathrm{e}.$$

Open discussion:
Testing the coupling between the potential and parallel momentum

- Minimal set of equations:

$$\begin{cases} \partial_t n + \nabla \cdot (nub + v_d) = S^n, \\ \partial_t (mnu) + \frac{1}{R} \nabla \cdot \big[Rmnu\big(ub + v_d\big) - R\nu \nabla u \big] = -\nabla_{\|} p + S_{\|}^M, \\ \nabla \cdot \big(\sigma \nabla \phi\big) = \nabla \cdot f(n, \nabla_{\|} p_{\mathrm{e}}, \nabla_{\|} T_{\mathrm{e}}, \nabla p), \end{cases}$$

with prescribed temperature profiles and

$$p_{\mathrm{e}} = n_{\mathrm{e}} T_{\mathrm{e}}, \qquad p_{\mathrm{i}} = n T_{\mathrm{i}}, \qquad p = p_{\mathrm{i}} + p_{\mathrm{e}},$$

$$v_d = c \frac{B \times \nabla \phi}{B^2} + c \frac{B \times \nabla p_{\mathrm{i}}}{ZenB^2} - \frac{D^n \nabla n}{n} - \frac{D^p \nabla p_{\mathrm{i}}}{n},$$

$$f = \sigma_{\|} \Big(\frac{\nabla_{\|} p_{\mathrm{e}}}{en_{\mathrm{e}}} + 0.71 \nabla_{\|} T_{\mathrm{e}} / e \Big) b + c \frac{B \times \nabla p}{B^2}.$$

Backup slides

Full system of Braginskii equations (FB)

Energy conservation

Proposition

A solution $n_\alpha, v_\alpha, p_\alpha, \phi$ is such that

$$\partial_t \mathcal{E} + \nabla \cdot \mathcal{F} = -\nabla\phi \cdot J + \sum_{\alpha \in \mathrm{Sp}} \big(S_{T,\alpha} + v_\alpha \cdot S_{M,\alpha} - \tfrac{1}{2} m_\alpha |v_\alpha|^2 S_{n,\alpha} \big),$$

where the energy/energy-flux pair is

$$\mathcal{E} = \sum_{\alpha \in \mathrm{Sp}} \big(\tfrac{1}{2} m_\alpha n_\alpha |v_\alpha|^2 + \tfrac{3}{2} p_\alpha \big), \; \mathcal{F} = \sum_{\alpha \in \mathrm{Sp}} \Big[\big(\tfrac{1}{2} m_\alpha n_\alpha |v_\alpha|^2 + \tfrac{5}{2} p_\alpha \big) v_\alpha + q_\alpha + \pi_\alpha \cdot v_\alpha \Big].$$

If $S_n = 0$, $S_M = 0$, $S_T = 0$ and the normal components $\mathcal{F}_n$ and J_n vanish on $\partial\Omega$,

$$\int_\Omega \mathcal{E}_0 dV = \sum_{\alpha \in \mathrm{Sp}} \int_\Omega \big(\tfrac{1}{2} m_\alpha n_\alpha |v_\alpha|^2 + \tfrac{3}{2} p_\alpha \big) dV = \mathrm{constant}.$$

Braginskii equations with zero electron mass (FB_0)

Energy conservation

Proposition

A solution $n_\alpha, v_\alpha, p_\alpha, \phi$ satisfies the energy balance

$$\partial_t \mathcal{E}_0 + \nabla \cdot \mathcal{F}_0 = -\nabla\phi \cdot J + sources$$

with energy/energy-flux pair

$$\mathcal{E}_0 = \sum_{a \in \mathrm{Sp}_0} \left(\tfrac{1}{2} m_a n_a |v_a|^2 + \tfrac{3}{2} p_a\right) + \tfrac{3}{2} p_\mathrm{e},$$

$$\mathcal{F}_0 = \sum_{a \in \mathrm{Sp}_0} \left[\left(\tfrac{1}{2} m_a n_a |v_a|^2 + \tfrac{5}{2} p_a\right) v_a + q_a + \pi_a \cdot v_a\right] + \tfrac{5}{2} p_\mathrm{e} v_\mathrm{e} + q_\mathrm{e}.$$

- Energy-conserving splitting schemes for this system have been derived by Juan Vicente Gutierrez-Santacreu.

Sketch of the proof of the first expression for J and v_{e}.

Parallel projection:

$$0 = -\nabla_{\|} p_{\mathrm{e}} + en_{\mathrm{e}} \nabla_{\|} \phi + en_{\mathrm{e}} J_{\|}/\sigma_{\|} + R_{T\|} + S_{M,\mathrm{e}\|},$$

which gives

$$J_{\|} = \sigma_{\|}\big[- \nabla\phi + E_{T\|}\big], \quad E_{T\|} = (\nabla_{\|} p_{\mathrm{e}} - R_{T\|} - S_{M,\mathrm{e}\|})/(en_{\mathrm{e}}).$$

The perpendicular projection:

$$(I - \kappa b\times) J_{\perp} = -\sigma_{\perp}\big[\nabla_{\perp}\phi + E_{T\perp}\big], \quad \kappa = \frac{\sigma_{\perp}|B|}{en_{\mathrm{e}}c},$$

where

$$E_{T\perp} = (\nabla_{\perp} p_{\mathrm{e}} - R_{T\perp} - S_{M,\mathrm{e}\perp} + \sum_{a\in \mathrm{Sp}_0} e_a n_a v_a \times B/c)/(en_{\mathrm{e}}).$$

The result follows from the fact that the matrix $(I - \kappa b\times)$ is invertible when restricted to vectors perpendicular to b. Also $(I - \kappa b\times)$ restricted to perpendicular vectors is positive-definite as $J_{\perp} \cdot (I - \kappa b\times) J_{\perp} = |J_{\perp}|^2$, hence so is its inverse. □

Sketch of the proof on the drift velocity.

Recall the definition (extended to multiple ion species with quasi-neutrality)

$$R_a = -e_a n_a \big(J_\| / \sigma_\| + J_\perp / \sigma_\perp\big) - R_T, \quad R_T = -0.71 n_a Z_a \nabla_\| T_\mathrm{e} - \frac{3 Z_a n_a}{2\omega_{ce}\tau_\mathrm{e}} b \times \nabla T_\mathrm{e}.$$

Multiply the momentum equation by $\frac{c}{e_a n_a B^2} B \times$ and solve for $v_{a\perp}$,

$$v_{a\perp} = c\frac{B \times \nabla\phi}{B^2} + c\frac{B \times \nabla p_a}{e_a n_a B^2} + \frac{3}{2\omega_{ce}\tau_\mathrm{e}}\frac{c}{e|B|}\nabla_\perp T_\mathrm{e} + c\frac{B \times J_\perp}{\sigma_\perp B^2} + \frac{c}{e_a n_a B^2} B \times \big[\partial_t(m_a n_a v_a) + \nabla\cdot(m_a n_a v_a \otimes v_a + \pi_a) - S_{M,a}\big].$$

Upon substituting the expression

$$J_\perp = \big[\hat{\sigma}(-\nabla\phi + E_T)\big]_\perp = \frac{c}{|B|} b \times \nabla p + \frac{c}{|B|} b \times \Big[\partial_t \mathcal{P} + \nabla\cdot\mathcal{S} - S_M\Big]$$

the diamagnetic current combines with the $\nabla_\perp T_\mathrm{e}$-term to give the classical diffusion, thus leaving $J_\perp^{(\mathrm{r})}$. The expression for D_a follows from $\sigma_\perp = n_\mathrm{e} e^2 \tau_\mathrm{e}/m_\mathrm{e}$, and the definition of partial temperature $T_a + Z_a T_\mathrm{e}$,

$$p = \sum_{\alpha\in\mathrm{Sp}} n_\alpha (k_B T_\alpha) = \sum_{a \in \mathrm{Sp}_0} n_a k_B (T_a + Z_a T_\mathrm{e}). \qquad \square$$