Proposal for ETS verification and benchmarking procedure

(Participation of ASTRA, CRONOS and JETTO is highly appreciated)

General remarks.

The test problems proposed below are considered as the next step after the series of verification tests of ETS based on the approach of manufactured solutions. Whereas the manufactured solutions serve to check overall accuracy of the numeric schemes the main goal of this series is to check
(1) implementation of particular terms in the transport set of equations,
(2) consistency and accuracy of coupling between the particular terms and equations in the set,
(3) conservation properties,
(4) stability,
(5) interface with equilibrium solvers.

Input parameters:

$R_0 = 5$ m, $B_0 = 5$ T,
$a_0 = 2$ m, $I_{pl} = 1 \div 10$ MA.

Notations and units:

All notations are based on a transport equation of the "density" form

$$\frac{\partial V' F}{\partial t} + \frac{\partial \Gamma}{\partial \rho} = V' s, \qquad 0 < \rho < a_0,$$

$$\Gamma|_{\rho=0}(t) = 0 \qquad \text{or equivalently} \qquad |\Gamma / S_{\mathrm{flux\_surf}}| < \infty.$$

Extension for energy-like quantity, i.e. (nF), is straightforward.

$0 \le \rho \le a_0$ – indepenedent variable (minor radius), [m],
$0 \le x = \rho/a_0 \le 1$ – indepenedent variable (normalized minor radius), [d/l],
$V(\rho, t)$ – plasma volume, [m^3], $(V'(\rho, t) = \partial V / \partial \rho)$, [m^2],
$S_\psi(\rho, t) = V' \langle |\nabla \rho| \rangle$ – flux surface area, [m^2],

$F(\rho, t)$ – generic dependent quantity, (any of ψ, $n_{e,i}$, $T_{e,i}$, $p_{e,i}$, etc.), [F],
$\Gamma(\rho, t) = V' \left\langle (\nabla \rho)^2 \right\rangle (vF - D\, \partial F / \partial \rho)$ – flux of the quantity F through the entire flux surface, [F m^3/s],
$D(\rho, t)$ – diffusion coefficient, [m^2/s]
$v(\rho, t)$ – convective velocity, [m/s]
$s(\rho, t)$ – source / sink of the quantity F, [F/s],
$S(\rho, t) = \int_0^\rho s\, V' d\rho$ – integrated source, [F m^3/s].

$P(A, B) = B + (A - B)(1 - (\rho/a_0)^2)$ – parabolic radial profile, with A being central and B edge values,
$H(X) = H(\rho/a_0 - X)$ – Heaviside function of radius.

N_ρ – number of radial grid points, [d/l]
τ – time step, [s]
N_τ – number of time steps, [d/l]

It is understood that density is multiplied by 10^{19} m^{-3}, temperature is given in keV while all other quantities are given in SI units)

Part I. Cylindrical geometry. Consistency check.

In this part:

$$V' = 4\pi R_0\rho, \quad \left\langle(\nabla\rho)^2\right\rangle \equiv 1, \quad \left\langle(\nabla\rho/R)^2\right\rangle \equiv R_0^{-2}, \quad \left\langle(R_0/R)^2\right\rangle \equiv 1, \quad V = 2\pi^2 R_0\rho^2, \quad q_{cyl} = \frac{5B_0a_0^2}{I_{pl}R_0}.$$

Number of “radial” grid points $N_\rho = \{20, 50, 100, 200, 500, 1000\}$,
Time step $\tau = \{\mathtt{1.E-1,1.E-2,1.E-3}\}$, if needed, e.g. for stiff problems, the time step can be reduced.

Characteristic times in the problems below vary by two orders of magnitude. Therefore, these grid parameters should be considered as reference values. It is expected that each test case runs with 2×2 different time and space steps in order to provide an accuracy estimate. If the implemented numerical scheme enables automatic time adjustment then the average time step for each problem and space grid would be very helpful for appraising the scheme.

Output: Total simulation time $t_{Stop} = \tau N_\tau = 10 \div 10^3$ s.
0D quantities should be given for all time points.

1D quantities (radial profiles) are expected for several $(5 \div 10)$ representative time slices.

Test I.1.1.

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	1 / 1	$P(2,1)$	–	0	0	0
$T_{i,1}$ / 2	1 / 1	$P(2,1)$	–	0	0	0
n_e / 2	1 / 1	$P(2,1)$	–	0	0	0
$n_{i,1}$ / 2	1 / 1	$P(2,1)$	–	0	0	0
$n_{i,2}$ / 0	–	–	–	–	–	–

Comment:
All F’s must stay unchanged.

Output: $n_{e,i}(\rho,t) - n_{e,i}(\rho,0)$, $T_{e,i}(\rho,t) - T_{e,i}(\rho,0)$.

Test I.1.2. Here $f(\rho,t) = 1 + \sin(t)$

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	1 / 1	$P(2,1)$	–	0	0	0
$T_{i,1}$ / 2	1 / 1	$P(2,1)$	–	0	0	0
n_e / 1	–	–	$P(f(\rho,t),1)$	0	0	0
$n_{i,1}$ / 1	–	–	$P(f(\rho,t),1)$	0	0	0
$n_{i,2}$ / 0	–	–	–	–	–	–

Comment:
$p_{e,i}$ must stay unchanged.

Output: $p_{e,i}(\rho,t) - p_{e,i}(\rho,0)$.

Test I.1.3.

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	4 / 0	$P(2,1)$	–	1	0	0
$T_{i,1}$ / 2	4 / 0	$P(2,1)$	–	1	0	0
n_e / 2	4 / 0	$P(2,1)$	–	1	0	0
$n_{i,1}$ / 2	4 / 0	$P(2,1)$	–	1	0	0
$n_{i,2}$ / 0	–	–	–	–	–	–

Comment:
Particles and energy must be conserved.

$$\int_V P(A,B)dV = \frac{A+B}{2}V,$$

$$\int_V P^2(A,B)dV = \frac{A^2-AB+B^2}{3}V$$

Exact solution for n is available

Output: Fluxes $\Gamma(\rho,t)$ and volume integrals $\int_0^\rho F\,V'd\rho$ for all Fs.

Test I.1.4. Similar to I.1.3 but the equipartition term is included on the rhs

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	4 / 0	$P(3,1)$	–	1	0	Q_{ie}
$T_{i,1}$ / 2	4 / 0	$P(1,1)$	–	1	0	Q_{ei}
n_e / 2	4 / 0	$P(2,1)$	–	1	0	0
$n_{i,1}$ / 2	4 / 0	$P(2,1)$	–	1	0	0
$n_{i,2}$ / 0	–	–	–	–	–	–

Comment:
Particles and energy must be conserved.

Output: Profiles, fluxes $\Gamma(\rho,t)$ and volume integrals $\int_0^\rho F\,V'd\rho$ for all Fs.

Test I.1.5. Similar to I.1.4 but a stepwise heating term is included

$$Q_{pulse}(\rho,t) = [H(\rho-\rho_1)-H(\rho-\rho_2)]\,[H(t-t_1)-H(t-t_2)]\,, \qquad 0<\rho_1<\rho_2<a_0, \quad 0<t_1<t_2<T$$

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	4 / 0	$P(3,1)$	–	1	0	$Q_{ie}+Q_{pulse}$
$T_{i,1}$ / 2	4 / 0	$P(1,1)$	–	1	0	Q_{ei}
n_e / 2	4 / 0	$P(2,1)$	–	1	0	0
$n_{i,1}$ / 2	4 / 0	$P(2,1)$	–	1	0	0
$n_{i,2}$ / 0	–	–	–	–	–	–

Output: Profiles, fluxes, volume integrals for Fs and $\int_0^T dt \int_0^{a_0} Q_{pulse}\,V'd\rho$.

Test I.1.6. Modulated heating term is included $Q_\omega = 1 + \sin\omega t$

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	1 / 0	$P(3,1)$	–	1	0	$Q_{ie}+Q_\omega$
$T_{i,1}$ / 2	1 / 0	$P(1,1)$	–	1	0	Q_{ei}
n_e / 1	–	–	$P(2,1)$	–	–	–
$n_{i,1}$ / 1	–	–	$P(2,1)$	–	–	–
$n_{i,2}$ / 0	–	–	–	–	–	–

Output: Fourier harmonics (in time) of T_e and T_i.

Test I.1.7.

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	4 / 0	$P(2,1)$	–	0.1	1	0
$T_{i,1}$ / 2	4 / 0	$P(2,1)$	–	0.1	1	0
n_e / 2	4 / 0	$P(2,1)$	–	0.1	1	0
$n_{i,1}$ / 2	4 / 0	$P(2,1)$	–	0.1	1	0
$n_{i,2}$ / 0	–	–	–	–	–	–

Test I.1.8.

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 1	–	–	–	–	–	–
T_e / 2	4 / 0	$P(2,1)$	–	0.1	-1	0
$T_{i,1}$ / 2	4 / 0	$P(2,1)$	–	0.1	-1	0
n_e / 2	4 / 0	$P(2,1)$	–	0.1	-1	0
$n_{i,1}$ / 2	4 / 0	$P(2,1)$	–	0.1	-1	0
$n_{i,2}$ / 0	–	–	–	–	–	–

At $D \to 0$ the equation degenerates so that the boundary condition at $\rho = 0$ cannot be satisfied. Nevertheless, it makes sense to push D in both examples down to zero in order to determine numeric limits and get an idea about residual numerical diffusion of the scheme.

For $v = \text{Const}$, equation $\frac{\partial}{\partial t}\rho n + \frac{\partial}{\partial \rho}\rho v n = 0$ has a general solution $n(\rho,t) = \frac{(\rho - vt)}{\rho} n_0(\rho - vt)$ where $n_0(\rho) = n(\rho, t=0)$.

Output: Profiles, fluxes $\Gamma(\rho,t)$ and volume integrals $\int\limits_0^\rho F\,V'd\rho$ for all Fs.

Comment:
For constant v and D the equation $\frac{\partial}{\partial t}\rho n + \frac{\partial}{\partial \rho}\rho\left(vn - D\frac{\partial n}{\partial \rho}\right) = 0$ has a steady state (asymptotic) solution $n(\rho,t) = Ce^{\frac{v}{D}\rho}$ where the constant C is defined by the particle conservation condition which for parabolic initial distribution $n(\rho, t=0) = P(n_0, n_1)$ gives

$$C = \frac{1}{a_0^2}\left[g\left(\frac{va_0}{D}\right)\right]^{-1}\int_0^{a_0} n(\rho,t=0)\rho\,d\rho = \frac{n_0+n_1}{4}\left[g\left(\frac{va_0}{D}\right)\right]^{-1},$$

with $g(x)$ being $g(x) = \left[1 + (x-1)e^x\right]/x^2$ and $g(x)|_{x\to 0} \approx \frac{1}{2} + \frac{x}{3}$, $g(x)|_{x\to\infty} \approx \frac{1}{x}e^x$.

Test I.1.10. "Poloidal" field energy dissipation. For $F = \psi$, $D(\rho, t)$ should be replaced by conductivity $\sigma_{\|}$.

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 2	2 / 10 MA	$P(0, 2\pi R_0)$	–	$\sigma_{\|}$	0	0
T_e / 2	1 / 1 keV	$P(2,1)$	–	1	0	$Q_{OH} + Q_{ie}$
$T_{i,1}$ / 2	1 / 1 keV	$P(2,1)$	–	1	0	Q_{ei}
n_e / 1	–	–	$P(2,1)$	1	0	0
$n_{i,1}$ / 1	–	–	$P(2,1)$	1	0	0

Comment: No steady state. Thermal and field energy must be conserved.

Output: Current density, loop voltage, safety factor, poloidal field energy, Poynting vector, Joule heating, energy contents as functions of time and radius.

Test I.1.11. Boundary condition – prescribed loop voltage.

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 2	3 / 0.2 V	$P(0, 2\pi R_0)$	–	$\sigma_{\|}$	0	0
T_e / 2	1 / 1 keV	$P(2,1)$	–	1	0	Q_{OH}
$T_{i,1}$ / 0	–	–	–	–	–	–
n_e / 1	–	–	$P(2,1)$	–	–	–
$n_{i,1}$ / 1	–	–	$P(2,1)$	–	–	–

Comment: Slow thermal instability can occur.

Output: Current density, loop voltage, poloidal field energy, Poynting vector, Joule heating.

Test I.1.12. Non-inductive current drive.

F value / flag	Bnd. type / value	$F(\rho,0)$	$F(\rho,t)$	D	v	s
ψ / 2	2 / 10 MA	$P(0, 2\pi R_0)$	–	$\sigma_{\|}$	0	(*)
T_e / 2	1 / 1	$P(2,1)$	–	1	0	Q_{OH}
$T_{i,1}$ / 0	–	–	–	–	–	–
n_e / 1	–	–	$P(2,1)$	–	–	–
$n_{i,1}$ / 1	–	–	$P(2,1)$	–	–	–

Comment: The total current should be replaced by a non-inductive current.

*) Noninductive current density is set to $j_{ni} = (\pi a)^{-2} \times 10^7$ A/m^2

Output: Poloidal field energy, Poynting vector, Joule heating, energy contents as functions of time and radius.

Part II. Stiff transport.

A simplified "cylindrical" diffusion equation for a quantity F reads

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{1}{x}\dfrac{\partial}{\partial x}\left(xD\dfrac{\partial F}{\partial x}\right) + S, \qquad 0 < x < 1, \qquad t > 0, \\ |DF_x(t, x=0)| < \infty, \qquad F(t, x=1) = F_1(t), \qquad F_x = \dfrac{\partial F}{\partial x}, \\ F(t=0, x) = F_0(x). \end{cases}$$

Diffusion coefficient is assumed to have the form $D = D_0 + D_{an}$.
A single equation for the main ion component n_i can be used here in place of F.
The estimate of accuracy should be based on the time behaviour of $F_x = \dfrac{\partial F}{\partial x}$ or D_{an} rather than $F(t, x)$.
It would be useful to have an output for the grid quantities

$$Q_f(t, x) = -xDF_x, \qquad Q_s(t, x) = \int_0^x xSdx, \qquad Q_t(t, x) = \frac{\partial}{\partial t}\int_0^x xFdx.$$

Test II.1. Simple model.
Stiff transport is described by $D_{an} = D_1 \max(0, -F_x - \eta_{cr})$ that switches on a stiff transport once $|F_x|$ exceeds η_{cr}. The following input parameters are proposed

$$\begin{array}{lll} F_0 = 0.1, & D_0 = 0.1, & \eta_{cr} = 1, \\ F_1 = 0.1, & D_1 = 1, & S = 1. \end{array}$$

Test II.2. Stiff transport + transport barrier.
$D_{an} = D_1 \min[\max(0, -F_x - \eta_{cr}), 0.07/(-F_x - \eta_{cr})$. The added correction suppresses the stiff transport in the range where $|F_x| > \eta_{cr} + \sqrt{0.07}$.
The input parameters are the same as in II.1 except for $S = 1 + P(1, 0)$. This change restricts an extension of the transport barrier to $0.525 < \rho/a_0 < 0.75$.

Test II.3. If a special scheme is implemented to treat stiff transport then two additional runs should be performed for a non-stiff transport $D_1 = 0$. One run should use the "stiff" numeric scheme, another a regular scheme. The aim is to evaluate distortions introduced by the stiff scheme to a non-stiff transport.

III. Toroidal geometry. (Coupling with equilibrium solver).

Tbd