Standardized Equations

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2009-01-30

Contents

1 General form of transport equation expected by numerical solver

$$\frac{a(\rho)\cdot n(\rho,t)-b(\rho)\cdot n(\rho,t-1)}{h}+\frac{1}{c(\rho)}\frac{\partial}{\partial\rho}\left(-d\left(\rho\right)\cdot\frac{\partial n(\rho,t)}{\partial\rho}+e\left(\rho\right)\cdot n\left(\rho,t\right)\right)= \\ f\left(\rho\right)-g\left(\rho\right)\cdot n\left(\rho,t\right) \tag{1}$$

with boundary conditions in the form:

$$v\left(\rho_{\mathrm{bnd}}\right)\cdot\left.\frac{\partial n\left(\rho,t\right)}{\partial\rho}\right|_{\mathrm{bnd}}+u\left(\rho_{\mathrm{bnd}}\right)\cdot n\left(\rho_{\mathrm{bnd}},t\right)=w\left(\rho_{\mathrm{bnd}}\right) \tag{2}$$

The following set of quantities is expected by the numerical solver :

$$\begin{aligned} & a_{\mathrm{mix}}\ h \\ & \rho \\ & a(\rho),\ b(\rho),\ c(\rho),\ d(\rho),\ e(\rho),\ f(\rho),\ g(\rho),\ n\left(\rho,t-1\right) \\ & u(1:2),\ v(1:2),\ w(1:2) \end{aligned} \tag{3}$$

taking in account:

$$v(1)=1,\ u(1)=0,\ w(1)=0$$

The solver will return:

$$n\left(\rho,t\right),\ \frac{\partial n\left(\rho,t\right)}{\partial\rho}$$

2 Translation to RITM numerical solver prepared for the DIFFERENTIAL form of equations

$$\frac{a\cdot n-b\cdot n^{-}}{h}+\frac{1}{c}\frac{\partial}{\partial\rho}\left(-d\cdot\frac{\partial n}{\partial\rho}+e\cdot n\right)=f-g\cdot n \tag{4}$$

$$n\cdot\left(\frac{a}{h}+g\right)+\frac{1}{c}\frac{\partial}{\partial\rho}\left(-d\cdot\frac{\partial n}{\partial\rho}+e\cdot n\right)=f\ +\ \frac{b\cdot n^{-}}{h} \tag{5}$$

$$\frac{1}{c}\left(d\cdot\frac{\partial^2 n}{\partial\rho^2}-e\cdot\frac{\partial n}{\partial\rho}+\frac{\partial d}{\partial\rho}\cdot\frac{\partial n}{\partial\rho}-\frac{\partial e}{\partial\rho}\cdot n\right)=n\cdot\left(\frac{a}{h}+g\right)-\left(f+\ \frac{b\cdot n^{-}}{h}\right) \tag{6}$$

$$\frac{\partial^2 n}{\partial\rho^2}-\frac{e}{d}\cdot\frac{\partial n}{\partial\rho}+\frac{1}{d}\frac{\partial d}{\partial\rho}\cdot\frac{\partial n}{\partial\rho}-\frac{1}{d}\frac{\partial e}{\partial\rho}\cdot n=n\cdot\frac{c}{d}\cdot\left(\frac{a}{h}+g\right)-\frac{c}{d}\cdot\left(f\ +\ \frac{b\cdot n^{-}}{h}\right) \tag{7}$$

$$\frac{\partial^2 n}{\partial\rho^2}+\frac{\partial n}{\partial\rho}\cdot\left(-\frac{e}{d}+\frac{1}{d}\frac{\partial d}{\partial\rho}\right)=n\cdot\frac{c}{d}\cdot\left(\frac{a}{h}+g+\frac{1}{c}\frac{\partial e}{\partial\rho}\right)-\frac{c}{d}\cdot\left(f\ +\ \frac{b\cdot n^{-}}{h}\right) \tag{8}$$

2.1 Equation required by numerical solver

$$\frac{\partial^2 n}{\partial \rho^2} + A \cdot \frac{\partial n}{\partial \rho} = B \cdot n - \quad C \tag{9}$$

with boundary conditions:

inner boundary

$$V\left(1\right) \cdot \frac{\partial n\left(1,t\right)}{\partial \rho} + U\left(1\right) \cdot n\left(1,t\right) = W\left(1\right) \tag{10}$$

outer boundary

$$V\left(2\right) \cdot \frac{\partial n\left(\mathrm{nrho},t\right)}{\partial \rho} + U\left(2\right) \cdot n\left(\mathrm{nrho},t\right) = W\left(2\right) \tag{11}$$

where $A\left(\rho\right)$, $B\left(\rho\right)$ and $C\left(\rho\right)$ are coefficient required by the RITM numerical solver for ordinary differential equation:

$$A\left(\rho\right) = -\frac{e\left(\rho\right)}{d\left(\rho\right)} + \frac{1}{d\left(\rho\right)}\frac{\partial d\left(\rho\right)}{\partial \rho} \tag{12}$$

$$B\left(\rho\right) = \frac{c\left(\rho\right) \cdot a\left(\rho\right)}{h} + \frac{c\left(\rho\right)}{d\left(\rho\right)} \cdot g\left(\rho\right) + \frac{1}{d\left(\rho\right)}\frac{\partial e\left(\rho\right)}{\partial \rho} \tag{13}$$

$$C\left(\rho\right) = \frac{c\left(\rho\right)}{d\left(\rho\right)} \cdot \left(f\left(\rho\right) \; + \; \frac{b\left(\rho\right) \cdot n\left(\rho, t-1\right)}{h}\right) \tag{14}$$

$V\left(1\right) = v\left(1\right)$	$U\left(1\right) = u\left(1\right)$	$W\left(1\right) = w\left(1\right)$
$V\left(2\right) = v\left(2\right)$	$U\left(2\right) = u\left(2\right)$	$W\left(2\right) = w\left(2\right)$

2.2 Outcome from RITM numerical solver

$Y\left(\rho\right)$ (*solution*) $\mathrm{DY}\left(\rho\right)$ (*derivative of solution*)

New function, and flux:

$$n_{\mathrm{new}}\left(\rho\right) = Y\left(\rho\right) \cdot a_{\mathrm{mix}} + n\left(\rho\right) \cdot \left(1 - a_{\mathrm{mix}}\right) \tag{15}$$

$$\frac{\partial n_{\mathrm{new}}\left(\rho\right)}{\partial \rho} = \mathrm{DY}\left(\rho\right) \cdot a_{\mathrm{mix}} + \frac{\partial n\left(\rho\right)}{\partial \rho} \cdot \left(1 - a_{\mathrm{mix}}\right) \tag{16}$$

3 Translation to RITM numerical solver prepared for the INTEGRAL form of equations

$$\frac{a \cdot n - b \cdot n^{-}}{h} + \frac{1}{c}\frac{\partial}{\partial \rho}\left(-d \cdot \frac{\partial n}{\partial \rho} + e \cdot n\right) = f - g \cdot n \tag{17}$$

$$n \cdot c \cdot \left(\frac{a}{h} + g\right) + \frac{\partial}{\partial \rho}\left(-d \cdot \frac{\partial n}{\partial \rho} + e \cdot n\right) = c \cdot \left(f \; + \; \frac{b \cdot n^{-}}{h}\right) \tag{18}$$

introduce new variables:

$$N = \frac{1}{\rho^2}\int_0^\rho n\, K\, \rho\, \partial\rho \tag{19}$$

$$K = \frac{c}{\rho}\cdot\left(\frac{a}{\tau}+g\right) \tag{20}$$

$$J = \int_0^\rho c\cdot\left(\frac{f}{\rho}+\frac{b\cdot n^-}{\rho\tau}\right)\rho\,\partial\rho \tag{21}$$

function, its gradient and flux:

$$n = \frac{\rho\,(\partial N/\partial\rho)+2N}{K} \tag{22}$$

$$\frac{\partial n}{\partial\rho} = \frac{\rho\left(\partial^2 N/\partial\rho^2\right)+3\ (\partial N/\partial\rho)}{K} - \frac{\rho\ (\partial N/\partial\rho)+2N}{K^2}\cdot\frac{\partial K}{\partial\rho} \tag{23}$$

$$\Gamma = J - N\cdot\rho^2 \tag{24}$$

equation to solve:

$$N\cdot\rho^2 - d\cdot\left(\frac{\rho\left(\partial^2 N/\partial\rho^2\right)+3\ (\partial N/\partial\rho)}{K} - \frac{\rho\ (\partial N/\partial\rho)+2N}{K^2}\cdot\frac{\partial K}{\partial\rho}\right) + e\cdot\frac{\rho\,(\partial N/\partial\rho)+2N}{K} = J \tag{25}$$

$$-N\frac{K\cdot\rho}{d} + \frac{\partial^2 N}{\partial\rho^2} + \frac{3}{\rho}\,\frac{\partial N}{\partial\rho} - \left(\frac{\partial N}{\partial\rho}+\frac{2}{\rho}N\right)\cdot\left(\frac{e}{d}+\frac{1}{K}\cdot\frac{\partial K}{\partial\rho}\right) = -\frac{\mathrm{JK}}{d\rho} \tag{26}$$

$$\frac{\partial^2 N}{\partial\rho^2} + \frac{\partial N}{\partial\rho}\cdot\left(\frac{3}{\rho}-\frac{e}{d}-\frac{1}{K}\cdot\frac{\partial K}{\partial\rho}\right) = N\cdot\left[\frac{K\cdot\rho}{d}+\frac{2}{\rho}\cdot\left(\frac{e}{d}+\frac{1}{K}\cdot\frac{\partial K}{\partial\rho}\right)\right] - \frac{\mathrm{JK}}{d\rho} \tag{27}$$

3.1 Equation required by numerical solver

$$\frac{\partial^2 N}{\partial\rho^2} + A\cdot\frac{\partial N}{\partial\rho} = B\cdot N - C \tag{28}$$

with boundary conditions:

inner boundary

$$V\,(1)\cdot\frac{\partial N\,(1,t)}{\partial\rho} + U\,(1)\cdot N\,(1,t) = W\,(1) \tag{29}$$

outer boundary

$$V\,(2)\cdot\frac{\partial N\,(\mathrm{nrho},t)}{\partial\rho} + U\,(2)\cdot N\,(\mathrm{nrho},t) = W\,(2) \tag{30}$$

where $A\,(\rho)$, $B\,(\rho)$ and $C\,(\rho)$ are coefficient required by the RITM numerical solver for ordinary differential equation:

$$
\begin{aligned}
A(\rho) &= \frac{3}{\rho} - \frac{e(\rho)}{d(\rho)} - \frac{1}{K(\rho)} \cdot \frac{\partial K(\rho)}{\partial \rho} && (31)\\
B(\rho) &= \frac{K(\rho)\cdot\rho}{d(\rho)} + \frac{2}{\rho}\cdot\left(\frac{e(\rho)}{d(\rho)} + \frac{1}{K(\rho)}\cdot\frac{\partial K(\rho)}{\partial\rho}\right) = \frac{K(\rho)\cdot\rho}{d(\rho)} - \frac{2}{\rho}\cdot A(\rho) + \frac{6}{\rho^2} && (32)\\
C(\rho) &= \frac{J(\rho)\cdot K(\rho)}{d(\rho)\cdot\rho} && (33)\\
V(1) &= \rho(1)\cdot(e(1)\cdot v(1) \ + d(1)\cdot u(1)) && (34)\\
U(1) &= v(1)\cdot\left[K(1)\cdot\rho(1)^2 + 2\cdot e(1)\right] + u(1)\cdot 2\cdot d(1) && (35)\\
W(1) &= K(1)\cdot(d(1)\cdot w(1) + J(1)\cdot v(1)) && (36)\\
V(2) &= \rho(\mathrm{nrho})\cdot(e(\mathrm{nrho})\cdot v(2) \ + d(\mathrm{nrho})\cdot u(2)) && (37)\\
U(2) &= v(2)\cdot\left[K(\mathrm{nrho})\cdot\rho(\mathrm{nrho})^2 + 2e(\mathrm{nrho})\right] + u(2)\cdot 2\cdot d(\mathrm{nrho}) && (38)\\
W(2) &= K(\mathrm{nrho})\cdot[d(\mathrm{nrho})\cdot w(2) + J(\mathrm{nrho})\cdot v(2)] && (39)
\end{aligned}
$$

3.2 Outcome from RITM numerical solver

$Y(\rho)$ (*solution*) $\mathrm{DY}(\rho)$ (*derivative of solution*)

New function, and flux:

$$
\begin{aligned}
n_{\mathrm{new}}(\rho) &= \frac{\rho\cdot\mathrm{DY}(\rho) + 2\cdot Y(\rho)}{K(\rho)}\cdot a_{\mathrm{mix}} + n(\rho)\cdot(1-a_{\mathrm{mix}}) && (40)\\
\frac{\partial n_{\mathrm{new}}(\rho)}{\partial\rho} &= \left(Y(\rho)\cdot\rho^2 + e(\rho)\,\frac{\rho\cdot\mathrm{DY}(\rho)+2\cdot Y(\rho)}{K(\rho)} - J(\rho)\right)\cdot\frac{a_{\mathrm{mix}}}{d(\rho)} + \frac{\partial n(\rho)}{\partial\rho}\cdot(1-a_{\mathrm{mix}}) && (41)
\end{aligned}
$$

4 Numerical solver prepared to solve the set of equations using "matrix PROGONKA"

The proposed method of solution ("matrix PROGONKA") can be applied to the wide class of the system of differential equations. The general form of the equations can be following:

$$
\begin{aligned}
&\frac{\tilde{a}^l(\rho)\cdot n^l(\rho,t) - \tilde{b}^l(\rho)\cdot n^l(\rho,t-\tau)}{\tau} + \sum_k \frac{1}{\tilde{c}^{\mathrm{lk}}(\rho)}\frac{\partial}{\partial\rho}\left(-\tilde{d}^{\mathrm{lk}}(\rho)\cdot\frac{\partial n^k(\rho,t)}{\partial\rho} + \tilde{e}^{\mathrm{lk}}(\rho)\cdot n^k(\rho,t)\right)\\
&= f^l(\rho) - \sum_k \tilde{g}^{\mathrm{lk}}(\rho)\cdot n^k(\rho,t)
\end{aligned}
\tag{42}
$$

The non diagonal term in matrices $\tilde{c}, \tilde{d}, \tilde{e}, \tilde{g}$ describes the coupling between equations. The finite difference approximation of the equations with respect to ρ leads to block tridiagonal matrix. The resulting algebraic equation can be cast into the form:

$$
\frac{a_s^l Y_s^l - b_s^l Y_s^l(t-\tau)}{\tau} = \sum_k (A_s^{\mathrm{lk}} Y_{s-1}^k + B_s^{\mathrm{lk}} Y_s^k + C_s^{\mathrm{lk}} Y_{s+1}^k) \tag{43}
$$

where $Y_s^l = Y^l(\rho_s)$ and s=1,2.…,Np. The algebraic equations can be solved using the block forward elimination and backward substitution method ("matrix PROGONKA".

For the ion density equations and the current equation the matrices $\tilde{c}, \tilde{d}, \tilde{e}, \tilde{g}$ can be directly expressed by a,b,c,d,e,f,g coefficients in Eq.(1). For these equation we have:

$$\begin{aligned} \tilde{c}^{\mathrm{lk}} &= c_l \delta_{\mathrm{lk}} & (44)\\ \tilde{d}^{\mathrm{lk}} &= d_l \delta_{\mathrm{lk}} & (45)\\ \tilde{e}^{\mathrm{lk}} &= e_l \delta_{\mathrm{lk}} & (46)\\ \tilde{g}^{\mathrm{lk}} &= g_l \delta_{\mathrm{lk}} & (47) \end{aligned}$$

where index l stands for equation number. Matrices A, B, C in equation 43 are diagonal and these equations can be solved one by one by standard forward elimination and backward substitution.

In the temperature equation there is the linear coupling between equations imbedded in the definition of $\mathrm{f_l}$. For the ion temperature equation the source term can be written as

$$f_j = V'[Q_{j,\mathrm{exp}} + \nu_{\mathrm{ei}} T_e + \sum_i v_{\mathrm{ji}} T_j] = f'_j + V' \nu_{\mathrm{ie}} T_e + V' \nu_{\mathrm{ji}} T_i$$

and for the electron temperature equation we can write

$$f = f' + \sum_i V' \nu_{\mathrm{ei}} T_i + \sum_i V' c_2 < |\nabla \rho|^2 > \frac{\Gamma_e}{n_e} \frac{\partial n_i}{\partial \rho} T_i + \sum_i V' c_2 < |\nabla \rho|^2 > \frac{\Gamma_e}{n_e} n_i \frac{\partial T_i}{\partial \rho}$$

The above equations define new value $\tilde{f}_i = f'_i$ and the no diagonal terms in the matrices $\tilde{g}^{\mathrm{ij}}$, $\tilde{c}^{\mathrm{ij}}$, $\tilde{e}^{\mathrm{ij}}$.

The equations for ion and electron temperatures can be solved simultaneously by "matrix progonka" method.

The toroidal velocity equations can also be solved together by "matrix progonka" method. The non diagonal term are defined by the equation:

$$f_i = f'_i + V' < R > \sum_{j \neq i} \frac{m_{\mathrm{ij}} n_j}{\tau_{\mathrm{ji}}} u_{j,\varphi}$$